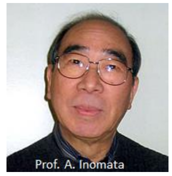
Favorite Physics Problems

by Andrew Ochadlick, Jr.

Classical Mechanics: From SUNYA's Prof. A. Inomata's Course



Phy 610 in the Fall of 1976

*In the process of solving some my favorite problems, the use of annoying phrases such as "we can write" and "one can readily verify", etc., that leaves the details to be resolved by Harold (Hypothetical alert reader of limitless dedication)**, *will be avoided as much as possible. This may result in too much detail for some readers and it is hoped that such readers appreciate the objective of making the solution transparent to all.*

**as found in Julian Schwinger's "Particles, Sources and Fields" of 1970*

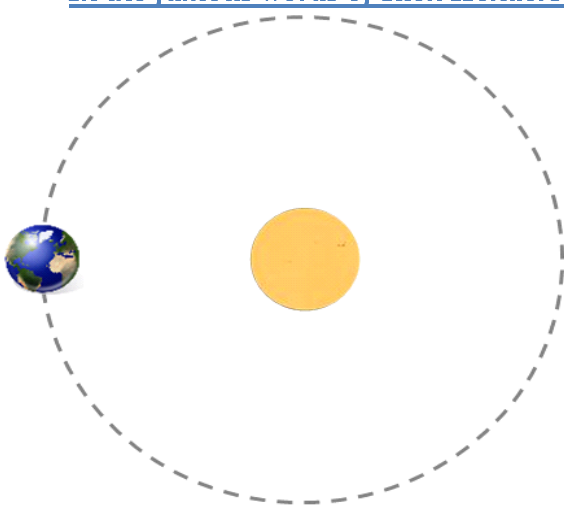
Classical Mechanics Problem: *Fall From Orbit*

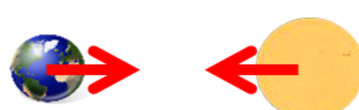
Two particles orbit one another in a circular orbit under the influence of gravitational forces, with a period τ. Their motion is suddenly stopped at a given instant of time, and they are released and allowed to fall into each other. Prove they collide after a time $\frac{\tau}{4\sqrt{2}}$.

(See Goldstein, Classical Mechanics, 7th printing, April 1965, problem #1 page 90.)

(note: 1AU= 1 Astronomical Unit = 1.496 10^{11} m =Mean distance between Earth and Sun.)

In the famous words of Rick Henderson: the following illustration is "not to scale".





Circular Orbital Motion Stopped, Objects Fall Into Each Other

Circular Orbital Motion

We'll consider the particles to be point particles to avoid bringing in the physical dimensions, of real objects; e.g., the radius of the Sun and the Earth and other unnecessary complications. Our

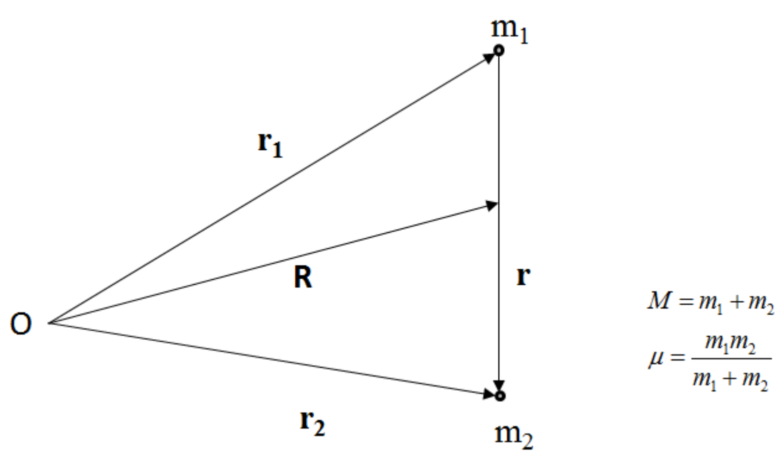
calculated time of fall into one another for the particles will involve a quantity which is the orbital period the particles undergo before the orbital motion is stopped. In order to incorporate the orbital period into our final result the orbital period, τ, will be derived first.
(Later, using the Earth and Sun, the velocity of fall of the Earth into the Sun will be plotted and fall time of the Earth into the Sun will also be plotted from the expressions developed below.)

For two orbiting particles, the Lagrangian is:

$$L = T - V = \frac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} + \frac{1}{2} m_2 \dot{\vec{r}}_2^{\,2} + G \frac{m_1 m_2}{|\vec{r}_1 - \vec{r}_2|}$$

Or in terms of the reduced mass

$$L = T - V = \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2} \mu \dot{\vec{r}}^{\,2} + G \frac{m_1 m_2}{|\vec{r}|}$$ where $M = m_1 + m_2$ and $\mu = \frac{m_1 m_2}{m_1 + m_2}$.



The form for the Lagrangian utilizing the reduced mass suggests that the reduced mass approach is simpler to implement. So, using the reduced mass, the equation of relative motion is $\mu \ddot{\vec{r}} = G \nabla \frac{m_1 m_2}{|\vec{r}|}$ or $\mu \ddot{r} - \mu r \dot{\theta}^2 = -G \frac{m_1 m_2}{r^2}$. This equation can be used to determine the orbital period. To begin the solution process, recognize that *r* is constant in a circular orbit. Since it is constant it is convenient to arbitrarily make the constant "*a*" so that $r = a$. Then with $\mu \ddot{r} = 0$ the equation of motion reduces to $\mu a \dot{\theta}^2 = G \frac{m_1 m_2}{a^2}$.

Solving for $\dot{\theta}$, the equation of motion becomes

$$\dot{\theta} = a^{-\frac{3}{2}} [G(m_1 + m_2)]^{\frac{1}{2}}.$$

Now, rewrite that equation as

$$\frac{d\theta}{dt} = a^{-\frac{3}{2}}\left[G(m_1 + m_2)\right]^{\frac{1}{2}}$$

and solve for dt to obtain

$$dt = \frac{a^{\frac{3}{2}}}{\left[G(m_1 + m_2)\right]^{\frac{1}{2}}}\, d\theta$$

Which can easily be integrated over one full revolution, i.e., $\theta = 2\pi$, which takes place in a time equal to the orbital period τ.

$$\int_0^{\tau} dt = \frac{a^{\frac{3}{2}}}{\left[G(m_1 + m_2)\right]^{\frac{1}{2}}} \int_0^{2\pi} d\theta$$

which reduces to $\tau = \dfrac{2\pi\, a^{\frac{3}{2}}}{\left[G(m_1 + m_2)\right]^{\frac{1}{2}}}$.

Now return to the main issue: if their motion is suddenly stopped at a given instant of time, and then they are released and allowed to fall into each other, the equation of motion, with the fact that the circular motion has stopped so that $\ddot{\theta} = 0$, becomes

$$\mu \ddot{r} = -G\frac{m_1 m_2}{r^2}.$$

To get this last equation in a form that can be integrated with respect to time, it is useful to recall that $\dfrac{d}{dt}(\dot{r}^2) = 2\dot{r}\ddot{r}$ which suggests that both sides of the previous equation can be multiplied by $\dfrac{2\dot{r}}{\mu}$ to provide the following solvable integral form:

$2\dot{r}\ddot{r} = -G\dfrac{m_1 m_2}{\mu}\dfrac{\dot{r}}{r^2}$. This can be rewritten as

$$\frac{d}{dt}(\dot{r}^2) = -G\frac{m_1 m_2}{\mu}\frac{\dot{r}}{r^2} \quad \text{or} \quad \frac{d}{dt}(\dot{r}^2) = -G\frac{m_1 m_2}{\mu}\frac{dr/dt}{r^2}.$$

Now multiply both sides by dt to obtain an expression that can be integrated:

$\int \frac{d}{dt}(\dot{r}^2)\,dt = -G\frac{m_1 m_2}{\mu}\int \frac{dr/dt}{r^2}\,dt$. The $dt's$ cancel to give the relatively simple integral

$\int d(\dot{r}^2) = -G\frac{m_1 m_2}{\mu}\int \frac{dr}{r^2}$. Then by using the expression for the reduced mass μ the result for the integrals becomes $\dot{r}^2 = 2G(m_1 + m_2)\frac{1}{r} + C$

Using the initial conditions that $\dot{r} = 0$ at $r = a$ determines the integration constant to be

$C = -\frac{2G}{a}(m_1 + m_2)$ so that $\dot{r}^2 = 2G(m_1 + m_2)(a - r)/ar$ which can be rewritten as

$dt = \left[\frac{2G}{a}(m_1 + m_2)\right]^{-1/2}\sqrt{\frac{r}{a-r}}\,dr$. With the particles starting to "fall" toward one another at an initial separation of $r = a$, the time for them to collide will be called time T and that occurs at $r = 0$. So the final integration is

$T = \int_0^T dt = \left[\frac{2G}{a}(m_1 + m_2)\right]^{-1/2}\int_a^0 \sqrt{\frac{r}{a-r}}\,dr$. The problem reduces to a solution to the integral: $\int_a^0 \sqrt{\frac{r}{a-r}}\,dr$.

To evaluate this integral, an approach using substitutions is given under the heading "**Appendix A**: Integral Solution via Substitutions" which can be seen at the end of this document. Using the result of Appendix A, the solution to the integral is:

$$T = \int_0^T dt = \left[\frac{2G}{a}(m_1 + m_2)\right]^{-1/2}\int_a^0 \sqrt{\frac{r}{a-r}}\,dr = \left[\frac{2G}{a}(m_1 + m_2)\right]^{-1/2}\left[-\sqrt{(a-r)r} - a\sin^{-1}\left(\frac{-2r+a}{a}\right)\right]_a^0$$

or

$$T = \frac{1}{4\sqrt{2}}\frac{2\pi a^{3/2}}{\sqrt{G(m_1 + m_2)}}.$$

(In that last step, T should be negative but we ignored the negative sign. It is actually negative because T is increasing and the distance is decreasing as the objects fall toward one another.)

Earlier the period of the circular motion was found to be $\tau = \dfrac{2\pi a^{\frac{3}{2}}}{\left[G(m_1+m_2)\right]^{\frac{1}{2}}}$ so that the final result for the time that it takes the particles to collide is $T = \dfrac{\tau}{4\sqrt{2}}$. And this last equation is precisely the expression the problem asked for.

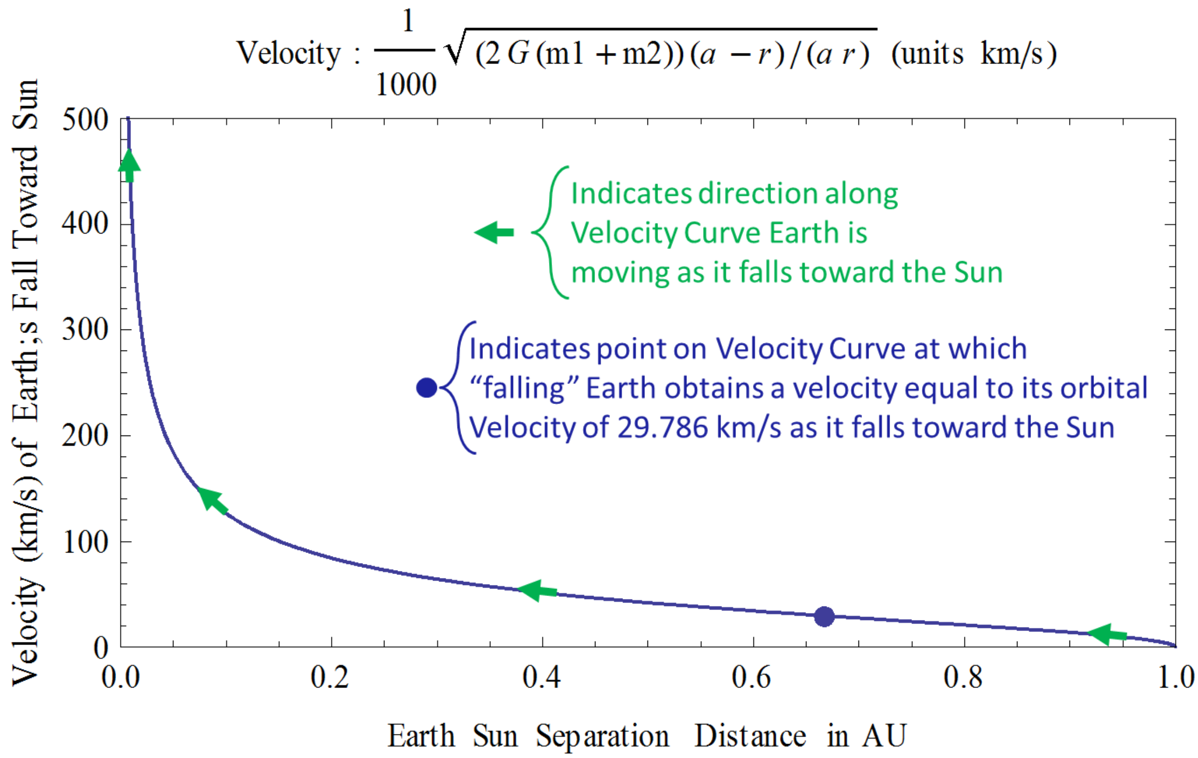
Application to an Approximation of the Earth-Sun System

Using parameters for a simple Earth Sun case, if the Earth stopped revolving about the Sun, and the Moon and other planets and other forces are ignored, then, _considering the Earth and Sun as points_, they would collide in

$$T = \frac{365.256\ days}{4\sqrt{2}} = 64.57\ \ days\ .$$

It is interesting to look into the problem a little further from the point of view of the velocity of fall and how the velocity varies as the particle separation decreases during the fall. To investigate the velocity, recall an equation from the above involving the velocity of fall, namely, $\dot{r}^2 = 2G(m_1+m_2)(a-r)/ar$.

And taking the square root gives the velocity as: $\dot{r} = \sqrt{2G(m_1+m_2)(a-r)/ar}$. Using that expression, the following figure plots the increase in the velocity of the fall of the Earth into the Sun as the separation during the fall decreases. The separation is shown in astronomical units (AU) where 1 AU represents the mean distance of the Earth from the Sun. The green arrows on the velocity curve, , represent the direction the Earth is moving on the velocity curve. (The single blue dot on the curve is explained later and further below.)



Here are some alternative and additional thoughts about this problem, including one that leads to a value that is close to the actual value of the time it takes for the Earth and the Sun to collide.

1.) If the Earth stopped revolving about the Sun, and the Moon and other planets and other forces (such as that from the solar wind) are ignored, then, considering the Earth and Sun as points, by the above result, with τ = 365.256 days, they would collide in

$$T = \frac{365.256\ days}{4\sqrt{2}} = 64.57\ \ days\ \ .$$

2.) Now as an approximation to this "exact" value, recall that while in its orbit the Earth is constantly falling toward the Sun as it orbits the Sun. In a sense, if a circular orbit is considered, then the Earth "falls" a distance of $2\pi R$ in one year or 365.256 days where R is the radius of an assumed circular orbit. So if the Earth stopped revolving about the Sun, you could approximate the time of fall into the Sun as $\frac{365.256 days}{2\pi} = 58.13\ days$.

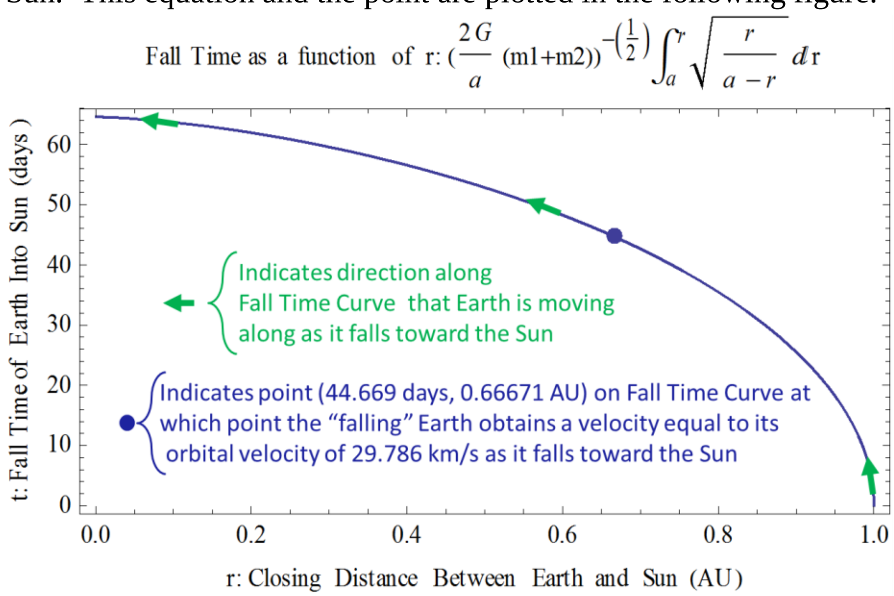
3.) At first it might seem surprising that the above estimate of 58.13 days gives a noticeably shorter time for the Earth to collide with the Sun which was calculated to take 64.57 days. However, the approximation giving 58.13 days does not have the Earth starting from zero velocity but has it moving at its orbital velocity all the way and that alters the approximate result to a smaller estimate even though the velocity of the fall in the exact solution increase as the two bodies get closer and closer to one another (as shown in Velocity Curve Figure), since, in the process, the acceleration is increasing.

4.) Following up on the above, how long does it take the falling Earth to reach the Earth's mean orbital velocity? The answer to that question might surprise some readers. For this Earth-Sun example, it is useful to recall that **the Earth's mean orbital velocity is 29.786 km/s**. Using $\dot{r} = \sqrt{2G(m_1+m_2)(a-r)/ar}$, for the problem of interest, the Earth reaches a velocity of 29.786 km/s from the Sun after it falls to within about 0.66671 AU. The blue dot, ●, in the above Velocity Curve represents the point with coordinates (29.786 km/s, 0.66671 AU) and, of course, for that dot on the velocity curve, the significant figures shown in the coordinates are more than necessary for a general discussion. From these values, the Earth has to fall approximately 1/3 of the original distance to the Sun before it acquires a velocity of 29.786 km/s.

5.) The next interesting question is how long does it take for the Earth to reach a distance of 0.66671 AU from the Sun? That is answered by taking the integral we had above, namely,

$$t = \left[\frac{2G}{a}(m_1+m_2)\right]^{-1/2} \int_a^r \sqrt{\frac{r}{a-r}}\,dr$$

and evaluating it with r = 0.66671 AU. Using the appropriate units, the time *t* (again ignoring sign) is found to be about 44.669 days for the Earth to reach a point at r = 0.66671 AU from the Sun. This equation and the point are plotted in the following figure.



And that concludes this discussion.

Appendix A: Integral Solution via Substitutions

$$\int \sqrt{\frac{x}{a-x}}dx = \int \frac{\sqrt{x}}{\sqrt{a-x}}dx$$

$$u=\sqrt{a-x} \;\; so\; u^2 = a-x \;\; or\, solving \;\; for \;\; x:\;\; x = a-u^2 \;\;\; or \;\; \sqrt{x}=\sqrt{a-u^2}$$

$$also \;\;\; from\; u=\sqrt{a-x} \;\;\; we\;\; get\;\; du = -\frac{1}{2}(a-x)^{-\frac{1}{2}}dx \;\; so\;\; dx = -2\sqrt{(a-x)}\; du$$

$$so\; \int \frac{\sqrt{x}}{\sqrt{a-x}}dx = \int \frac{\sqrt{a-u^2}}{\sqrt{a-x}}(-2)\sqrt{(a-x)}\; du$$

$$=-2\int \sqrt{a-u^2}\; du, \;\; now \;\; let \;\; u=\sqrt{a}\sin\theta \;\; then \;\; du = \sqrt{a}\cos\theta\, d\theta$$

$$=-2\int \sqrt{a-a\sin^2\theta}\;\sqrt{a}\cos\theta\, d\theta$$

$$=-2a\int \cos\theta\cos\theta\, d\theta = -2a\int \cos^2\theta\, d\theta$$

$$recall \;\; that \;\; \cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$$

$$=-a\int(1+\cos 2\theta)$$

$$=-a\theta - a\int \cos 2\theta d\theta$$

$$let\;\; \phi = 2\theta, \; then \;\; d\theta = \frac{1}{2}d\phi$$

$$-a\theta - a\int \frac{\cos\phi}{2}d\phi \;\;\; = -a(\theta + \frac{1}{2}\sin\phi)$$

$$recall \quad \sin\theta = \frac{u}{\sqrt{a}} \;\; so \;\; \theta = \sin^{-1}\left(\frac{u}{\sqrt{a}}\right) = \sin^{-1}\left(\frac{\sqrt{a-x}}{\sqrt{a}}\right) = \sin^{-1}\sqrt{\frac{a-x}{a}}$$

$$so - a(\theta + \frac{1}{2}\sin\phi) = -a\left(\sin^{-1}\sqrt{\frac{a-x}{a}} + \frac{1}{2}\sin\phi\right)$$

$we\ had\ \phi = 2\theta\ then$

$$\sin\phi = \sin 2\theta = 2\sin\theta\cos\theta = 2\sin\theta\sqrt{1-\sin^2\theta}$$

$$= 2\frac{u}{\sqrt{a}}\sqrt{1-\frac{u^2}{a}} = 2\sqrt{\frac{a-x}{a}}\sqrt{1-\frac{a-x}{a}} = 2\sqrt{\frac{a-x}{a}}\sqrt{\frac{x}{a}} = \frac{2}{a}\sqrt{(a-x)x}$$

$recalling\ \ we\ \ had\ \ -a\left(\sin^{-1}\sqrt{\frac{a-x}{a}}+\frac{1}{2}\sin\phi\right)\ and\ \ with\ \ the\ above$

$$then\ -a\left(\sin^{-1}\sqrt{\frac{a-x}{a}}+\frac{1}{2}\sin\phi\right) = -a\left(\sin^{-1}\sqrt{\frac{a-x}{a}}+\frac{1}{2}\frac{2}{a}\sqrt{(a-x)x}\right)$$

$$= -a\left(\sin^{-1}\sqrt{\frac{a-x}{a}}+\frac{1}{2}\frac{2}{a}\sqrt{(a-x)x}\right) = -a\sin^{-1}\sqrt{\frac{a-x}{a}}-\sqrt{(a-x)x}$$

Therefore:

$$\int\sqrt{\frac{x}{a-x}}dx = -a\sin^{-1}\sqrt{\frac{a-x}{a}}-\sqrt{(a-x)x}$$